

Some new moment rearrangement invariant spaces; theory and applications.

EUGENE OSTROVSKY, LEONID SIROTA

*Department of Mathematic, HADAS company,
56209, Hertzelia Pituach, Galgaley Aplada street, 16, ISRAEL;
e - mail: galo@list.ru
Bar - Ilan University, 59200, Ramat Gan, ISRAEL;
e - mail: sirita@zahav.net.il*

Abstract.

In this article we introduce and investigate some new Banach spaces, so - called moment spaces, and consider applications to the Fourier series, singular integral operators, theory of martingales.

Key words: *Banach, Orlicz, Lorentz, Marcinkiewicz, moment and rearrangement invariant spaces, martingales, singular operators, Fourier series and transform.*

Mathematics Subject Classification. Primary (1991) 37B30, 33K55, Secondary (2000) 34A34, 65M20, 42B25.

1 Definitions. Simple Properties.

Let (X, Σ, μ) be a measurable space with non - trivial measure $\mu : \exists A \in \Sigma, \mu(A) \in (0, \mu(X))$. We will assume that either $\mu(X) = 1$, or $\mu(X) = \infty$ and that the measure μ is σ -finite and diffuse: $\forall A \in \Sigma, 0 < \mu(A) < \infty \exists B \subset A, \mu(B) = \mu(A)/2$. Define as usually for all the measurable function $f : X \rightarrow R^1$

$$|f|_p = \left(\int_X |f(x)|^p \mu(dx) \right)^{1/p}, \quad p \geq 1;$$

$L_p = L(p) = L(p; X, \mu) = \{f, |f|_p < \infty\}$. Let $a = \text{const} \geq 1, b = \text{const} \in (a, \infty]$, and let $\psi = \psi(p)$ be some positive continuous on the *open* interval (a, b) function, such that there exists a measurable function $f : X \rightarrow R$ for which

$$\psi(p) = |f|_p, \quad p \in (a, b).$$

Note that the function $p \rightarrow p \cdot \log \psi(p)$, $p \in (a, b)$ is convex.

The set of all those functions we will denote $\Psi : \Psi = \Psi(a, b) = \{\psi(\cdot)\}$. We can describe all those functions.

Theorem 0. *Let the measure μ is diffuse. The function $\nu(p)$, $p \in (a, b)$ belongs to the set Ψ if and only if there exist a two functions $\Lambda_1(p)$, $\Lambda_2(p)$, such that $\nu^p(p) = \Lambda_1(p) + \Lambda_2(p)$, where $\Lambda_1(p)$ is absolute monotonic on the interval (a, b) and $\Lambda_2(p)$ is relative monotonic on the interval $(a, b) : \forall k = 0, 1, 2, \dots$*

$$\forall p \in (a, b) \Rightarrow \Lambda_1^{(k)}(p) \geq 0, \quad (-1)^k \Lambda_2^{(k)}(p) \geq 0.$$

Proof. Let $\nu(\cdot) \in \Psi$, then $\exists f : X \rightarrow R$, $\nu^p(p) =$

$$\int_X |f(x)|^p \mu(dx) = \int_X \exp(p \log |f(x)|) \mu(dx) = \Lambda_1(p) + \Lambda_2(p),$$

where

$$\Lambda_1(p) = \int_{\{x:|f(x)| \geq 1\}} \exp(p \log |f(x)|) \mu(dx), \quad \Lambda_1^{(k)}(p) \geq 0;$$

$$\Lambda_2(p) = \int_{\{x:|f(x)| < 1\}} \exp(p \log |f(x)|) \mu(dx), \quad (-1)^k \Lambda_2^{(k)}(p) \geq 0.$$

Inversely, assume that $\nu^p(p) = \Lambda_1(p) + \Lambda_2(p)$, $\Lambda_1^{(k)}(p) \geq 0$, $(-1)^k \Lambda^{(k)}(p) \geq 0$. It follows from Bernstein's theorem that

$$\Lambda_1(p) = \int_R \exp(pt) \mu_1(dt), \quad \Lambda_2(p) = \int_R \exp(pt) \mu_2(dt),$$

where μ_1, μ_2 are a Borel measures on the set R such that $\text{supp } \mu_1 \in [0, \infty)$, $\text{supp } \mu_2 \in (-\infty, 0]$ and

$$\forall p \in (a, b) \Rightarrow \Lambda_1(p) < \infty, \quad \Lambda_2(p) < \infty.$$

Therefore

$$\nu^p(p) = \int_{-\infty}^{\infty} \exp(pt) (\mu_1(dt) + \mu_2(dt)).$$

Since the measure μ is diffuse, there exists a (measurable) function $\eta : X \rightarrow R$ such that

$$\nu^p(p) = \int_X \exp(p\eta(x)) \mu(dx).$$

Thus, for $f(x) = \exp(\eta(x))$ we obtain:

$$|f|_p^p = \int_X \exp(p\eta(x)) \mu(dx) = \nu^p(p), \quad |f|_p = \nu(p).$$

Corollary 1. Note that if $\psi_1(\cdot) \in \Psi$, $\psi_2(\cdot) \in \Psi$, then $\psi_1(\cdot) \cdot \psi_2(\cdot) \in \Psi$. Indeed, if

$$\psi_1(p) = |f_1|_p, \quad \psi_2(p) = |f_2|_p,$$

and the functions f_1, f_2 are independent, then

$$\psi_1(p) \cdot \psi_2(p) = |f_1 \cdot f_2|_p.$$

We extend the set Ψ as follows:

$$EX\Psi \stackrel{\text{def}}{=} EX\Psi(a, b) = \{\nu = \nu(p)\} =$$

$$\{\nu : \exists \psi(\cdot) \in \Psi : 0 < \inf_{p \in (a, b)} \psi(p)/\nu(p) \leq \sup_{p \in (a, b)} \psi(p)/\nu(p) < \infty\},$$

$$U\Psi \stackrel{\text{def}}{=} U\Psi(a, b) = \{\psi = \psi(p), \forall p \in (a, b) \Rightarrow \psi(p) > 0\}$$

and the function $p \rightarrow \psi(p)$, $p \in (a, b)$ is continuous.

Hereafter $a = \text{const} \geq 1$, $b \in (a, \infty]$.

Since the case $\psi(a+0) < \infty$, $\psi(b-0) < \infty$ is trivial for us, we will assume further that either $\psi(a+0) = \infty$ or $\psi(b-0) = \infty$, or both the cases: $\psi(a+0) = \psi(b-0) = \infty$.

We define in the case $b = \infty$ $\psi(b-0) = \lim_{p \rightarrow \infty} \psi(p)$.

Definition 1. Let $\psi(\cdot) \in U\Psi(a, b)$. The space $G(\psi) = G(X, \psi) = G(X, \psi, \mu) = G(X, \psi, \mu, a, b)$ consist on all the measurable functions $f : X \rightarrow R$ with finite norm

$$\|f\|_{G(\psi)} \stackrel{\text{def}}{=} \sup_{p \in (a, b)} [|f|_p / \psi(p)].$$

The spaces $G(\psi)$, $\psi \in U\Psi$ are non - trivial: arbitrary bounded $\sup_x |f(x)| < \infty$ measurable function $f : X \rightarrow R$ with finite support: $\mu(\text{supp } |f|) < \infty$ belongs to arbitrary space $G(\psi)$, $\forall \psi \in U\Psi$.

We denote as usually $\text{supp } \psi = \{p : \psi(p) < \infty\}$.

Now we consider a very important for applications examples of $G(\psi)$ spaces. Let $a = \text{const} \geq 1, b = \text{const} \in (a, \infty]; \alpha, \beta = \text{const}$. Assume also that at $b < \infty$ $\min(\alpha, \beta) \geq 0$ and denote by h the (unique) root of equation

$$(h - a)^\alpha = (b - h)^\beta, \quad a < h < b; \quad \zeta(p) = \zeta(a, b; \alpha, \beta; p) =$$

$$(p - a)^\alpha, \quad p \in (a, h); \quad \zeta(a, b; \alpha, \beta; p) = (b - p)^\beta, \quad p \in [h, b);$$

and in the case $b = \infty$ assume that $\alpha \geq 0, \beta < 0$; denote by h the (unique) root of equation $(h - a)^\alpha = h^\beta, h > a$; define in this case

$$\zeta(p) = \zeta(a, b; \alpha, \beta; p) = (p - a)^\alpha, \quad p \in (a, h); \quad p \geq h \Rightarrow \zeta(p) = p^\beta.$$

Note that at $b = \infty \Rightarrow \zeta(p) \asymp (p - a)^\alpha p^{-\alpha+\beta} \asymp \min\{(p - a)^\alpha, p^\beta\}, p \in (a, \infty)$ and that at $b < \infty \Rightarrow \zeta(p) \asymp (p - a)^\alpha (b - p)^\beta \asymp \min\{(p - a)^\alpha, (b - p)^\beta\}, p \in (a, b)$. Here and further $p \in (a, b) \Rightarrow \psi(p) \asymp \nu(p)$ denotes that

$$0 < \inf_{p \in (a, b)} \psi(p)/\nu(p) \leq \sup_{p \in (a, b)} \psi(p)/\nu(p) < \infty.$$

We will denote also by the symbols $C_j, j \geq 1$ some "constructive" finite non - essentially positive constants. By definition, $I(A) = I(A, x) = I(x \in A) = 1, x \in A; I(A) = 0, x \notin A$.

Definition 2. The space $G = G_X = G_X(a, b; \alpha, \beta) = G(a, b; \alpha, \beta)$ consists on all measurable functions $f : X \rightarrow R^1$ with finite norm

$$\|f\|G(a, b; \alpha, \beta) = \sup_{p \in (a, b)} [|f|_p \cdot \zeta(a, b; \alpha, \beta; p)].$$

Corollary 2. As long as the cases $\alpha \leq 0; b < \infty, \beta \leq 0$ and $b = \infty, \beta \geq 0$ are trivial, we will assume further that either $1 \leq a < b < \infty, \min(\alpha, \beta) > 0$, or $1 \leq a, b = \infty, \alpha \geq 0, \beta < 0$.

Lemma 1. Let $\psi \in U\Psi$, $\psi(a + 0) = \psi(b - 0) = \infty, b < \infty$. There exist a two functions $\nu_1, \nu_2 \in U\Psi, \nu_1(a + 0) \in (0, \infty), \nu_1(p) \sim \psi(p), p \rightarrow b - 0; \nu_2(b - 0) \in (0, \infty), \nu_2(p) \sim \psi(p), p \rightarrow a + 0$ such that the space $G(\psi)$ may be represented as a direct sum

$$G(\psi) = G(\nu_1) + G(\nu_2).$$

Proof. Indeed, if $f = f_1 + f_2, f_1 \in G(\nu_1), \nu_1 \in U\Psi, \nu_1(a + 0) \in (0, \infty); f_2 \in G(\nu_2), \nu_2 \in U\Psi, \nu_2(b - 0) \in (0, \infty)$, then $f_1 \in G(\psi), f_2 \in G(\psi)$, hence $f \in G(\psi)$.

Inversely, let $\psi \in U\Psi, \psi(a + 0) = \psi(b - 0) = 0$. Let p_0 be a some number inside the interval (a, b) such that

$$\psi(p_0) = \min_{p \in (a, b)} \psi(p) \stackrel{\text{def}}{=} C.$$

Define

$$\nu_1(p) = \psi(p) \cdot I(p \in (a, p_0)) + C \cdot I(p \in [p_0, b)),$$

$$\nu_2(p) = C \cdot I(p \in (a, p_0)) + \psi(p) \cdot I(p \in [p_0, b)).$$

If $f \in G(\psi)$, then

$$f(x) = f(x)I(|f(x)| \geq 1) + f(x)I(|f(x)| < 1) = f_1 + f_2,$$

where by virtue of Tchebychev's inequality: $\mu\{x : |f(x)| \geq 1\} < |f|_p < \infty$ for some $p \in (a, b)$ it follows that $f_1 \in G(\nu_1)$; and since $\forall q > p, A \in \Sigma$

$$\int_A |f_2|^q \mu(dx) \leq \int_A |f_2|^p \mu(dx),$$

we obtain $f_2 \in G(\nu_2)$.

It is evident by virtue of Liapunov's inequality that in the bounded case $\mu(X) = 1 : G(\psi) = G(\nu_1)$.

We denote by $G^o = G_X^o(\psi)$, $\psi \in U\Psi$ the closed subspace of $G(\psi)$, consisting on all the functions f , satisfying the following condition:

$$\lim_{p \rightarrow a+0} |f|_p / \psi(p) = \lim_{p \rightarrow b-0} |f|_p / \psi(p) = 0,$$

in the case $\psi(a+0) = \infty$, $\psi(b-0) = \infty$;

$$\lim_{p \rightarrow b-0} |f|_p / \psi(p) = 0$$

in the case $\psi(a+0) < \infty$, $\psi(b-0) = \infty$;

$$\lim_{p \rightarrow a+0} |f|_p / \psi(p) = 0$$

in the case $\psi(a+0) = \infty$, $\psi(b-0) < \infty$; and by $GB = GB(\psi)$ the closed span in the norm $G(\psi)$ the set of all the bounded measurable functions with finite support: $\mu(\text{supp } |f|) < \infty$.

Another definition: for a two functions $\nu_1(\cdot)$, $\nu_2(\cdot) \in U\Psi$ we will write $\nu_1 << \nu_2$, iff

$$\lim_{p \rightarrow a+0} \nu_1(p) / \nu_2(p) = \lim_{p \rightarrow b-0} \nu_1(p) / \nu_2(p) = 0$$

in the case $\nu_2(a+0) = \nu_2(b-0) = \infty$ etc.

If for some $\nu_1(\cdot), \nu_2(\cdot) \in U\Psi$, $\nu_1 << \nu_2$ and $\|f\|G(\nu_1) < \infty$, then $f \in G^o(\nu_2)$. Moreover, if there exists a sequence of functions f_n, f_∞ such that for some $\nu_1 \in G(\psi, a, b)$

$$\forall p \in (a, b) \Rightarrow |f_n - f_\infty|_p \rightarrow 0, n \rightarrow \infty$$

and $\sup_{n \leq \infty} \|f_n\|G(\nu_2) < \infty$, then $\|f_n - f_\infty\|G(\nu_1) \rightarrow 0$.

We consider now some important examples. Let $X = R$, $\mu(dx) = dx, 1 \leq a < b < \infty, \gamma = \text{const} > -1/a, \nu = \text{const} > -1/b, p \in (a, b)$,

$$f_{a,\gamma} = f_{a,\gamma}(x) = I(|x| \geq 1) \cdot |x|^{-1/a} (|\log |x||)^\gamma,$$

$$g_{b,\nu} = g_{b,\nu}(x) = I(|x| < 1) \cdot |x|^{-1/b} |\log x|^\nu,$$

$$h_m(x) = (\log |x|)^{1/m} I(|x| < 1), m = \text{const} > 0,$$

$$f_{a,b;\gamma,\nu}(x) = f_{a,\gamma}(x) + g_{b,\nu}(x), g_{a,\gamma,m}(x) = h_m(x) + f_{a,\gamma}(x),$$

$$\psi_{a,b;\gamma,\nu}^p(p) = 2(1 - p/b)^{-p\nu-1} \Gamma(p\gamma + 1) + 2(p/a - 1)^{-p\gamma-1} \Gamma(p\nu + 1),$$

$$\psi_{a,\gamma,m}^p(x) = 2(p/a - 1)^{-p\gamma-1} \Gamma(p\gamma + 1) + 2\Gamma((p/m) + 1),$$

$\Gamma(\cdot)$ is usually Gamma - function.

We find by the direct calculation:

$$|f_{a,b;\gamma,\nu}|_p^p = \psi_{a,b;\gamma,\nu}^p(p); \quad |g_{a,\gamma,m}|_p^p = \psi_{a,\gamma,m}^p(p).$$

Therefore,

$$\psi_{a,b;\gamma,\nu}(\cdot) \in \Psi(a, b), \quad \psi_{a,\gamma,m}(\cdot) \in \Psi(a, \infty).$$

Further,

$$f_{a,b;\gamma,\nu}(\cdot) \in G(a, b; \gamma + 1/a, \nu + 1/b) \setminus G^o(a, b; \gamma + 1/a, \nu + 1/b),$$

$$g_{a,\gamma,m}(\cdot) \in G \setminus G^o(a, \infty; \gamma + 1/a, -1/m),$$

and $\forall \Delta \in (0, 1)$ $f_{a,b,\alpha,\beta} \notin$

$$G(a, b; (1 - \Delta)(\gamma + 1/a), \nu + 1/b) \cup G(a, b; 1/a, (1 - \Delta)(\nu + 1/b),$$

$$g_{a,\gamma,m}(\cdot) \in G \setminus G^o(a, \infty; \gamma + 1/a; -1/m).$$

Another examples. Put

$$f^{(a,b;\alpha,\beta)}(x) = |x|^{-1/b} \exp \left(C_1 |\log x|^{1-\alpha} \right) I(|x| < 1) +$$

$$I(|x| \geq 1) |x|^{1/a} \exp \left(C_2 (\log x)^{1-\beta} \right);$$

$1 \leq a < b < \infty; \alpha, \beta = \text{const} \in (0, 1)$. We have:

$$\log \left| f^{(a,b;\alpha,\beta)}(\cdot) \right|_p \asymp (p - a)^{1-1/\alpha} + (b - p)^{1-1/\beta}, \quad p \in (a, b).$$

Theorem 1. *The spaces $G(\psi)$ with respect to the ordinary operations and introduced norm $\|\cdot\|_{G(\psi)}$ are Banach spaces.*

We need only to prove the completeness of $G(\psi)$ - spaces. Denote

$$\epsilon(n, m) = \|f_n - f_m\|_{G(\psi)}, \quad \epsilon(n) = \sup_{m \geq n} \epsilon(m, n),$$

and assume that $\lim_{n,m \rightarrow \infty} \epsilon(m, n) = 0$; then $\lim_{n \rightarrow \infty} \epsilon(n) = 0$. Let $p(i), i = 1, 2, \dots$ be the (countable) sequence of *all* rational numbers of interval (a, b) . We have from the direct definition of our spaces:

$$\forall p \in (a, b) \Rightarrow |f_n - f_m|_{p(i)} \leq \epsilon(n, m) \psi(p(i)).$$

As long as the spaces $L(p(i))$ are complete, we conclude that there exist functions $f^{(i)}, f^{(i)} \in L(p(i))$ such that

$$|f_n - f^{(i)}|_{p(i)} \leq \epsilon(n) \psi(p(i)) \rightarrow 0, \quad n \rightarrow \infty.$$

It is evident that

$$\mu\{x : \forall i \ f^{(i)}(x) \neq f^{(1)}(x)\} = 0,$$

i.e. $f^{(i)}(x) = f^{(1)}(x)$ μ -almost everywhere. Hence $\forall i = 1, 2, \dots$

$$|f_n - f^{(1)}|_{p(i)} \leq \epsilon(n) \psi(p(i)),$$

$$\begin{aligned} \forall p \in (a, b) &\Rightarrow |f_n - f^{(1)}|_p \leq \epsilon(n)\psi(p), \\ ||f_n - f^{(1)}||G(\psi) &= \sup_{p \in (a, b)} |f_n - f^{(1)}|_p / \psi(p) \leq \epsilon(n) \rightarrow 0, \end{aligned}$$

$n \rightarrow \infty$. This completes the proof of theorem 1.

Moreover, the spaces $G(\cdot)$ are rearrangement invariant (r.i.) spaces with the fundamental function

$$\phi(G, \delta) \stackrel{def}{=} ||I(A)||G, \quad A \in \Sigma, \quad \mu(A) = \delta \in (0, \infty).$$

In our case, for the spaces $G(\psi)$, $\psi(\cdot) \in U\Psi$, $\text{supp } \psi = (a, b)$, $b \leq \infty$ we have:

$$\phi(G(\psi), \delta) = \sup_{p \in (a, b)} [\delta^{1/p} / \psi(p)].$$

Note that in the case $b < \infty$

$$\delta \leq 1 \Rightarrow C_1 \delta^{1/a} \leq \phi(G, \delta) \leq C_2 \delta^{1/b},$$

$$\delta > 1 \Rightarrow C_3 \delta^{1/b} \leq \phi(G, \delta) \leq C_4 \delta^{1/a}.$$

Moreover, $\lambda \in (0, 1) \Rightarrow$

$$\lambda^{1/b} \phi(G, \delta) \leq \phi(G, \lambda \delta) \leq \lambda^{1/a} \phi(G, \delta);$$

$$\lambda > 1 \Rightarrow \lambda^{1/b} \phi(G, \delta) \leq \phi(G, \lambda \delta) \leq \lambda^{1/a} \phi(G, \delta).$$

For instance, define in the case $b < \infty$ $\delta_1 = \exp(\alpha h^2 / (h - a))$, $\delta \geq \delta_1 \Rightarrow$

$$p_1 = p_1(\delta) = \log \delta / (2\alpha) - [0.25\alpha^{-2} \log^2 \delta - a\alpha^{-1} \log \delta]^{1/2},$$

$$\phi_1(\delta) = \delta^{1/p_1} (p_1 - a)^\alpha;$$

$$\delta \in (0, \delta_1) \Rightarrow \phi_1(\delta) = \delta^{1/h} (h - a)^\alpha;$$

$$\delta_2 = \exp(-h^2 \beta / (b - h)), \quad \delta \in (0, \delta_2) \Rightarrow$$

$$p_2 = p_2(\delta) = -|\log \delta| / 2\beta + [\log^2(\delta / (4\beta^2)) + b|\log \delta| / \beta]^{1/2},$$

$$\phi_2(\delta) = \delta^{1/p_2(\delta)} (b - p_2(\delta))^\beta;$$

$$\delta \geq \delta_2 \Rightarrow \phi_2(\delta) = \delta^{1/h} (b - h)^\beta.$$

We obtain after some calculations:

$$b < \infty \Rightarrow \phi(G(a, b; \alpha, \beta), \delta) = \max[\phi_1(\delta), \phi_2(\delta)].$$

Note that as $\delta \rightarrow 0+$

$$\phi(G(a, b, \alpha, \beta), \delta) \sim (\beta b^2 / e)^\beta \delta^{1/b} |\log \delta|^{-\beta},$$

and as $\delta \rightarrow \infty$

$$\phi(G(a, b, \alpha, \beta), \delta) \sim (a^2 \alpha / e)^\alpha \delta^{1/a} (\log \delta)^{-\alpha}.$$

In the case $b = \infty, \beta < 0$ we have denoting

$$\phi_3(\delta) = (\beta / e)^\beta |\log \delta|^{-|\beta|}, \quad \delta \in (0, \exp(-h|\beta|)),$$

$$\phi_3(\delta) = h^{-|\beta|} \delta^{1/h}, \quad \delta \geq \exp(-h|\beta|) :$$

$$\phi(G(a, \infty; \alpha, -\beta), \delta) = \max(\phi_1(\delta), \phi_3(\delta)),$$

and we receive as $\delta \rightarrow 0+$ and as $\delta \rightarrow \infty$ correspondingly:

$$\phi(G(a, \infty; \alpha, -\beta), \delta) \sim (\beta)^{|\beta|} |\log \delta|^{-|\beta|},$$

$$\phi(G(a, \infty; \alpha, -\beta), \delta) \sim (a^2 \alpha / e)^\alpha \delta^{1/a} (\log \delta)^{-a}.$$

2 Connection with another r.i. spaces.

Theorem 2. A. Let $\psi(\cdot) \in EX\Psi$, such that $\exists g : X \rightarrow R$, $\psi(p) \asymp |g(\cdot)|_p$, $p \in (a, b)$. Denote

$$N^{(-1)}(1/\delta) = 1/(\phi(G(\psi), \delta)), \quad \delta \in (0, \infty),$$

where $N^{(-1)}$ denotes the left inverse function to the $N(\cdot)$ on the set R_+ . If

$$\forall \epsilon > 0 \quad \int_X N(\epsilon |g(x)|) \mu(dx) = \infty, \quad (2.1)$$

then the space $G(\psi)$ is not equivalent to arbitrary Orlicz's space $Or(X, \mu, \Phi)$.

B. Denote $T(x) = (1/\phi(x))^{(-1)}$. If

$$\sup_{p \in \text{supp } \psi} \left[\left(\int_0^\infty x^{p-1} T(x) dx \right) / \psi(p) \right]^{1/p} = \infty, \quad (2.2)$$

then the space $G(\psi)$ is not equivalent to arbitrary Marcinkiewicz's space $M(\theta)$.

C. Let $\psi(\cdot) \in U\Psi$, $\text{supp } \psi = (a, b)$, $1 \leq a < b < \infty$. Then the space $G(\psi)$ is not equivalent to arbitrary Lorentz space $L(\chi)$.

Proof. A. Assume conversely, i.e. that $G(\psi) \sim Or(\Phi)$, where $Or(\Phi)$ is some Orlicz's space on the set (X, Σ, μ) with corresponding (convex, even, $\Phi(0) = 0$ etc.) Orlicz's function $\Phi(u)$, $u \in R$. Since for $A \in \Sigma$, $\mu(A) \in (0, \infty)$

$$\phi(Or(\Phi); \mu(A)) = \|I(A)\|_{Or(\Phi)} = 1 / [\Phi^{-1}(1/\mu(A))],$$

we conclude that $\Phi(u) = N(u)$. It is evident that $g(\cdot) \in G(\psi) = Or(\Phi)$, but $g(\cdot) \notin Or(\Phi)$ by virtue of our condition (2.1). This contradiction proves the assertion **A**.

As a consequence:

Lemma 2. The space $G(a, b; \alpha, \beta)$ are equivalent to the Orlicz's space *only in the case* $\alpha = 0, b = \infty, \beta < 0$.

(The case $\alpha = 0, b = \infty, \beta < 0$ was considered in [12].)

Proof B. Assume conversely, i.e. that the space $G(\psi) = G(\psi, a, b)$ is equivalent to some Marcinkiewicz space $M(\theta)$ over the our measurable space (X, μ) . Recall here that in the considered case $a \geq 1; b > a$ the norm of a function $f : X \rightarrow R$ in the Marcinkiewicz space may be calculated by the formula

$$\|f\|_{M(\theta)} = \sup_{\delta > 0} [\theta(\delta) T^{(-1)}(f, \delta)]$$

and that the fundamental function for the $M(\theta)$ space is equal to

$$\phi(M(\theta), \delta) = 1/\theta(\delta),$$

(see, for example, [21], p. 187). Therefore, if the space $G(\psi)$ is equivalent to some Marcinkiewicz space $M(\theta)$, then

$$\theta(\delta) = \delta / \phi(G(\psi), \delta).$$

Let us consider the function $f : X \rightarrow R$ with the tail - function $T(f, x) = T(x)$, then $f \in M(\theta)$, but it follows from our condition (2.2) that $f \notin G(\psi)$.

For example, all the spaces $G(a, b; \alpha, \beta)$ are not equivalent to arbitrary Marcinkiewicz space.

Proof C is very simple, again by means of the method of "reduction in absurdum". Suppose $G(\psi) \sim L(\chi)$, where $L(\chi)$ denotes the Lorentz space with some (quasi) - concave generating function $\chi(\cdot)$. Since

$$\phi(L(\chi), \delta) = \chi(\delta) \rightarrow 0, \delta \rightarrow 0+$$

and $\chi(\delta) \rightarrow \infty, \delta \rightarrow \infty$, we conclude that the space $L(\chi) = G(\psi)$ is separable ([22], p. 150.) But we will prove further (in the section 4) that the space $G(\psi)$ are non - separable.

3 Norm's absolute continuity.

We will say that the function $f \in G(\psi)$, $\psi \in U\Psi$ has *absolute continue norm* and write $f \in GA(\psi)$, if

$$\lim_{\delta \rightarrow 0} \sup_{A: \mu(A) \leq \delta} \|f I_A\|_{G(\psi)} = 0.$$

The subspaces $GA(\psi), GB(\psi), G^0(\psi)$ are closed subspaces of space $G(\psi)$.

Theorem 3. Let $\psi \in U\Psi$. Then

$$G^0(\psi) = GB(\psi) = GA(\psi).$$

For example, if $\min(\alpha, |\beta|) > 0, 1 \leq a < b \leq \infty$, then

$$G^o(a, b; \alpha, \beta) = GB(a, b; \alpha, \beta) = GA(a, b; \alpha, \beta).$$

Proof. The inclusions $GB \subset GA, GA \subset G^o$ are obvious. Let now $f \in G^0$; for simplicity we will suppose $b < \infty, \mu(X) = 1$. Then $\lim_{p \rightarrow b-0} |f|_p / \psi(p) = 0$. Let $\epsilon > 0$. We have: $\|f I(|f| \geq N)\|_G \leq$

$$\sup_{p \in [1, b-\delta]} |f I(|f| \geq N)|_p / \psi(p) + \sup_{p \in (b-\delta, b)} |f|_p / \psi(p) = \Sigma_1 + \Sigma_2;$$

$$\Sigma_2 \leq \sup_{p \in [b-\delta, b)} |f|_p / \psi(p) \leq \epsilon/2$$

for some $\delta \in (0, b)$ by virtue of condition $f \in G^o$.

Further, there exists a value $N \geq 1$ such that

$$\Sigma_1 \leq C |f I(|f| \geq N)|_{b-\delta} \leq \epsilon/2$$

as long as $f \in L_{b-\delta}$. Following, $f \in GB$; thus $G^0 \subset GB$.

Now we prove the inverse embedding. Let $f \in GB, \epsilon > 0$. Then $\exists g, \sup_x |g(x)| = B < \infty, \forall p \in [1, b) \Rightarrow |f - g|_p / \psi(p) < \epsilon/2$,

$$|f|_p \leq |g|_p + 0.5\epsilon\psi(p), \quad p \in [1, b);$$

$$|f|_p / \psi(p) \leq |g|_p / \psi(p) + 0.5\epsilon < 0.5\epsilon + 0.5\epsilon \leq \epsilon, \quad |p - b| < \delta$$

for sufficiently small value δ . Theorem 3 is proved.

We investigate here the *sufficient* condition for the convergence

$$\|f_n - f_\infty\|G(\psi) \rightarrow 0, \quad n \rightarrow \infty. \quad (3.1)$$

Assume at first that (the necessary condition)

$$\mathbf{A}. \forall p \in (a, b) \quad |f_n - f_\infty|_p \rightarrow 0, \quad n \rightarrow \infty.$$

Theorem 4. Let $f_n, f_\infty \in G(\psi)$. Assume that (in addition to the condition **A**)

B. $\exists \psi_2(\cdot) \in U\Psi, \psi < \psi_2$, such that

$$\sup_{n \leq \infty} \|f_n\|G(\psi_2) < \infty.$$

Then the convergence (3.1) holds.

Proof. We need use the following auxiliary well - known facts.

1. Let $1 \leq a < b \in (1, \infty)$. We assert that

$$\sup_{p \in (a, b)} |f|_p < \infty \Leftrightarrow \max(|f|_a, |f|_b) < \infty.$$

This proposition follows from the formula

$$|f|_p^p = p \int_0^\infty z^{p-1} T(f, z) dz,$$

Tchebychev's inequality and Fatou's lemma.

2. Let $1 \leq p(1) \leq p \leq p(2) < \infty, \max(|f|_{p(1)}, |f|_{p(2)}) < \infty$. Then $|f|_p \leq$

$$|f|_{p(1)}^{(p(2)-p)/(p(2)-p(1))} \cdot |f|_{p(2)}^{(p-p(1))/(p(2)-p(1))} \stackrel{def}{=} Z(p, p(1), p(2); |f|_{p(1)}, |f|_{p(2)}).$$

Proposition 2 follows from Hölder's inequality.

It is sufficient to investigate the case $b < \infty$; another cases may be proved analogously. Consider the norm

$$\Sigma \stackrel{def}{=} \|f_n - f_\infty\|G(\psi) = \sup_{p \in (a, b)} |f_n - f_\infty|_p / \psi(p).$$

Let $\epsilon = const > 0$. We have: $\Sigma \leq \Sigma_1 + \Sigma_2 + \Sigma_3$, where $\Sigma_1 =$

$$\sup_{p \in (a, a+\delta)} |f_n - f_\infty|_p / \psi(p) \leq$$

$$\sup_p [|f_n - f_\infty|_p / \psi_2(p)] \cdot \sup_{p \in (a, a+\delta)} \psi(p) / \psi_2(p) \leq C(a, \delta) < \epsilon/3,$$

if $\delta = \delta(\epsilon)$ is sufficiently small. Further, $\Sigma_3 =$

$$\sup_{p \in (b-\delta, \delta)} [|f_n - f_\infty|_p / \psi_2(p)] \cdot \sup_{p \in (b-\delta, b)} [\psi(p) / \psi_2(p)] \leq C(b, \delta) < \epsilon/3.$$

Finally, $\Sigma_2 \leq$

$$\sup_{p \in (a+\delta, b-\delta)} |f|_p / \psi(p) \leq CZ(p, a+\delta, b-\delta, |f_n - f_\infty|_{a+\delta}, |f_n - f_\infty|_{b-\delta})$$

$< \epsilon/3$ for sufficiently large values n .

Analogously may be proved the following assertion about the $G(\psi)$ convergence.

Lemma 3. *If the sequence of a functions $\{f_n(\cdot)\}$ convergens in all the L_p norms:*

$$\forall p \in (a, b) \Rightarrow \lim_{n \rightarrow \infty} |f_n - f_\infty|_p = 0$$

and has a uniform absolute continuous norms in the $G(\psi)$ space:

$$\lim_{\delta \rightarrow 0+} \sup_{n \leq \infty} \sup_{A: \mu(A) \leq \delta} ||f_n I(A)||G(\psi) = 0,$$

then $||f_n - f_\infty||G(\psi) \rightarrow 0, n \rightarrow \infty$.

Theorem 5. *Let $\psi \in U\Psi$. We assert that $||f||G/G^o = ||f||G/GA =$*

$$||f||G/GB = \inf_{g \in GB} ||f - g||G = \overline{\lim}_{\delta \rightarrow 0+} \sup_{A: \mu(A) \leq \delta} ||f I(A)||G.$$

Here the notation G/G^o denotes the factor - space.

Proof. Suppose for simplicity $b \in (1, \infty)$, $\mu(X) = 1, G = G(\psi), \psi(a+0) < \infty, \psi(b-0) = \infty; f \in G \setminus G^o$. Put

$$\gamma = \overline{\lim}_{\delta \rightarrow 0} \sup_{A: \mu(A) \leq \delta} ||f I(A)||G > 0.$$

Let also $g = g(x)$ be a measurable bounded function: $\sup_x |g(x)| = B \in (0, \infty); k = \text{const} \geq 2$. We conclude using the elementary inequality: $X \geq kY > 0, k > 2, Y \leq B = \text{const} \Rightarrow$

$$\frac{(X - Y)^p}{X^p - B^p} \geq \frac{(k - 1)^p}{k^p - 1} :$$

$$||f - g||G \geq \sup_{p \in [1, b)} \left[\int_{\{x: |f(x)| > k|g(x)|\}} |f(x) - g(x)|^p \mu(dx) \right]^{1/p} / \psi(p) \geq$$

$$\overline{\lim}_{p \rightarrow b-0} \left[\int_{\{|f(x)| \geq kB\}} (k - 1)^p (k^p - 1)^{-1} (|f|^p - B^p) \mu(dx) \right]^{1/p} / \psi(p) \geq$$

$$(k - 1)(k^b - 1)^{-1/b} \overline{\lim}_{\delta \rightarrow 0} ||f I(A)||G = (k - 1)(k^b - 1)^{-1/b} \gamma.$$

Since the value of k is arbitrary, it follows from the last inequality that $||f - g||G \geq \gamma$; this proves that $\inf_{g \in GB} ||f - g||G \geq \gamma$; the inverse inequality is evident.

4 Non - separability.

Recall that $\psi(a+0) = \infty$ or $\psi(b-0) = \infty$.

Theorem 6. *The spaces $G(\psi), \psi \in U\Psi$ are non - separable.*

Proof. The assertion of theorem 6 is trivial if the metric space $(\Sigma, \rho(A, B)), \rho(A, B) = \arctan(\mu(A \Delta B))$ is non - separable. Therefore by virtue of Rockling's theorem we can suppose the space X is equipped by the distance

$d = d(x_1, x_2)$ such that the space (X, d) is complete and separable, the measure μ is Borelian and diffuse.

Conversely, assume that the space $G(\psi)$ is separable. Let $\{u_n(x)\}$ be a enumerable dense subset of $G(\psi)$. By virtue of Lusin's and Prokhorov's theorems we conclude that there exists a compact subset Y of X with $\mu(Y) > 0$ such that on the subspace Y all the functions $u_n(x)$ are continuous. We consider now the space $G(Y, \psi)$. The functions $\{u_n(x)\}, x \in Y$ belong to the space $G_Y^o(\psi)$. Let $w(x), x \in Y$, be some function from the space $G_Y(\psi) \setminus G_Y^o(\psi)$ and define $w(x) = 0, x \in X \setminus Y$. We get:

$$\inf_n \|w - u_n\|_{G_X} \geq \inf_n \|w - u_n\|_{G_Y} \geq \inf_{g \in GB_Y} \|w - g\|_{G_Y} > 0,$$

in contradiction. This completes the proof of theorem 3.

Our proof of theorem 3 is the same as proof of non - separability of Orlicz's spaces ([1], p. 103; [2], p. 127).

5 Adjoint spaces.

The complete description of spaces conjugated to $\cap_p L_p$, see in [3], [4]. The spaces which are conjugate to Orlicz's spaces are described in [2], p. 128 - 132. The structure of spaces $G^*(\psi)$ is analogous.

It is easy to verify that the structure of linear continuous functionals over the space $G^0(\psi) = GA = GB$ is follows: $\forall l \in G^{0*}(\psi) \Rightarrow \exists g : X \rightarrow R$,

$$l(f) = \int_X f(x)g(x) \mu(dx).$$

We investigate here only some necessary conditions for the inclusion $g \in G^*(\psi)$. Notation: $l_g(f) = \int_X f(x)g(x)\mu(dx)$. Note at first that if $\psi \in U\Psi(a, b)$, $q \in (b/(b-1), a/(a-1))$ and $g \in L_q$, then $g \in G^*(\psi)$.

Theorem 7. *If $g \in G^*$, then $\exists K = K(g) < \infty \Rightarrow$*

$$\forall z > 0 \Rightarrow \int_z^\infty T(g, u)du \leq K\phi(G, T(g, z)).$$

Proof. Let $l_g \in G^*$. It follows from uniform boundedness principle that $\forall f \in G \Rightarrow$

$$|l_g(f)| = \left| \int_X f(x) g(x)\mu(dx) \right| \leq K\|f\|_G.$$

Put $f = I_A(x)$, $A \in \Sigma$, $A = \{x : |g(x)| > z\}$, $z > 0$; then

$$\int_z^\infty T(g, u)du = \int_X |g(x)|I(|g(x)| > z) \mu(dx) \leq K\phi_G(T(g, z)).$$

Let now $\psi \in U\Psi$, $\text{supp } \psi = (a, b)$, $b < \infty$. Introduce the following N - Orlicz function

$$N_\psi(u) = \sup_{p \in (a, b)} \left[|u|^p \psi^{-p}(p) \right],$$

then the following implication holds:

$$\exists \epsilon > 0 \int_X N_\psi(\epsilon f)\mu(dx) < \infty \Rightarrow f \in G(\psi).$$

Therefore, the Orlicz's space $Or(N, X, \mu)$ is subspace of $G(\psi)$. Following,

$$(G(\psi))^* \subset (L(N_\psi))^*.$$

Since the function $N_\psi(u)$ satisfies the Δ_2 condition, the adjoint space $(L(H_\psi))^*$ may be described as a new Orlicz's space, namely

$$(L(N_\psi))^* = L(\Phi_\psi), \quad \Phi_\psi(u) = \sup_{z \in R} (uz - N_\psi).$$

Thus, we obtained: $\psi \in U\Psi(a, b), 1 \leq a < b < \infty \Rightarrow$

$$(G(\psi))^* \subset L(\Phi_\psi).$$

6 Tail behavior.

Let $f \in G(\psi), \psi \in U\Psi(a, b), b \leq \infty$. It follows from Tchebychev's inequality that

$$T(f, u) \leq \inf_{p \in (a, b)} [||f||^p \psi(p) / u^p], \quad u > 0.$$

Conversely,

$$|f|_p^p = p \int_0^\infty u^{p-1} T(f, u) du, \quad p \geq 1;$$

therefore

$$||f||_{G(\psi)} = \sup_{p \in \text{supp } \psi} \left[p \left[\int_0^\infty u^{p-1} T(f, u) du \right]^{1/p} / \psi(p) \right].$$

In the particular case the spaces $G(a, b; \alpha, \beta)$ we obtain after simple calculations:

Theorem 8. A. *Let $f \in G(a, b; \alpha, \beta), 1 \leq a < b < \infty$. Then*

$$u \in (0, 1/2) \Rightarrow T(f, u) \leq C_1(a, b, \alpha, \beta) |\log u|^{\alpha\alpha} u^{-a}; \quad (5.1)$$

$$u \geq 2 \Rightarrow T(f, u) \leq C_2(a, b, \alpha, \beta) (\log u)^{b\beta} u^{-b}. \quad (5.2)$$

B. *Conversely, suppose $\exists a, b, 1 \leq a < b < \infty, \gamma, \tau \geq 0, C_j > 0$ such that*

$$T(f, u) \leq C_1 |\log u|^\gamma u^{-a}, \quad u \in (0, 1/2); \quad T(f, u) \leq C_2 (\log u)^\tau u^{-b}, \quad u \geq 2.$$

Then $f \in G(a, b; \gamma + 1, \tau + 1)$.

C. *Let now $f \in G(a, \infty; \alpha, -\beta), \beta > 0$. We propose that*

$$T(f, u) \leq C_1 |\log u|^{\alpha\alpha} u^{-a}, \quad u \in (0, 1/2],$$

$$T(f, u) \leq C_2 \exp(-C_3 u^{1/\beta}), \quad u \geq 1/2;$$

D. *Conversely, if $\exists a \geq 1, \beta > 0, \gamma \geq 0$,*

$$T(f, u) \leq C_1 |\log u|^\gamma u^{-a}, \quad u \in (0, 1/2), a = \text{const} > 0, \gamma \geq 0,$$

$$T(f, u) \leq C_2 \exp(-C_3 u^{1/\beta}), \quad \beta > 0,$$

then $f \in G(a, \infty; \gamma + 1, -\beta)$.

Note in addition that at $\min(\alpha, \beta) > 0, b < \infty$

$$T(f, u) \sim C_1 |\log u|^{\alpha\alpha} u^{-a}, \quad u \rightarrow 0+ \Leftrightarrow |f|_p \sim C_2 (p - a)^{-\alpha}, \quad p \rightarrow a + 0;$$

$$T(f, u) \sim C_3 |\log u|^{b\beta} u^{-b}, u \rightarrow \infty \Leftrightarrow |f|_p \sim C_4 (b-p)^{-\beta}, p \rightarrow b-0$$

(Richter's theorem).

Despite the well - known Richter's theorem, we can show that both the inequalities (5.1) and (5.2) are exact. Let us consider the following examples.

Example 5.1. Let $\mu(X) = 1$, i.e. (X, Σ, μ) is a probability space and let μ is diffuse. Consider the (measurable) discrete = valued function $f : X \rightarrow R$ such that

$$\mu\{x : f(x) = \exp(\exp(k))\} = C \exp(\beta b k - b \exp k), k = 1, 2, \dots;$$

$$1/C = \sum_{k=1}^{\infty} \exp(\beta b k - b \exp(k)),$$

and denote $\gamma = \beta b$, $a(k) = a(k, \gamma, \epsilon) = \exp(k\gamma - \epsilon \exp(k))$,

$$\epsilon = b - p \rightarrow 0+, k(0) \stackrel{def}{=} [\log(\gamma/\epsilon)], x(k) = \exp(\exp(k)),$$

here $[z]$ denotes the integer part of z . We get:

$$W(\epsilon) \stackrel{def}{=} C^{-1} |f|_p^p = \sum_{k=1}^{\infty} a(k, \gamma, \epsilon) \geq$$

$$C_2 a(k(0), \gamma, \epsilon) \geq C_3 (b-p)^{-b\beta},$$

therefore $|f|_p \geq C_4 (b-p)^{-\beta}$.

On the other hand, we have at $k > k(0)$ and $k < k(0)$ correspondently

$$a(k+1)/a(k) < \exp(\gamma(e-2)) < 1, a(k-1)/a(k) < \exp(-\gamma/e) < 1,$$

hence

$$W(\epsilon) \leq C_3 a(k(0), \gamma, \epsilon) \leq C \epsilon^{-p\beta},$$

following $|f|_p \leq C_5 (b-p)^{-\beta}, p \in (1, b)$. Thus $f \in G(1, b; 0, \beta)$. However,

$$T(|f|, x(k)) > C \exp(b\beta k - b \exp k) = C (\log x(k))^{b\beta} x(k)^{-b}.$$

(we used the discrete analog of saddle - point method).

Example 5.2. Let $X = R_+^1, \mu(dx) = dx, Q(k) = \exp(a\alpha k + a \exp(k)), a = const \geq 1, S(k) = \sum_{l=1}^k Q(l), b \in (a, \infty)$,

$$g(x) = \sum_{k=1}^{\infty} \exp(-\exp(k)) I(x \in (S(k-1), S(k)]),$$

$u(k) = \exp(-\exp(k))$. We obtain analogously to the example 5.1:

$$p \in (a, b) \Rightarrow |g|_p \asymp (p-a)^{-\alpha},$$

but

$$T(g, u(k)) \geq C(a, b, \alpha) |\log u(k)|^{a\alpha} u(k)^{-a}.$$

7 Fourier's transform.

In this section we investigate the boundedness of certain Fourier's operators, convergence and divergence Fourier's series and transforms in $G(\psi)$ spaces. Let $X = [-\pi, \pi]$ or $X = R = (-\infty, \infty)$, $\mu(dx) = dx$, $c(n) = c(n, f) =$

$$\int_{-\pi}^{\pi} \exp(inx) f(x) dx, n = 0, \pm 1, \pm 2 \dots; 2\pi s_M[f](x) =$$

$$\sum_{\{n: |n| \leq M\}} c(n) \exp(-inx), s^*[f] = \sup_{M \geq 1} |s_M[f]|,$$

$$F[f](x) = \lim_{M \rightarrow \infty} \int_{-M}^M \exp(itx) f(t) dt,$$

$$F^*[f](x) = \sup_{M > 0} \int_{-M}^M \exp(itx) f(t) dt,$$

$$S_M[f](x) = (2\pi)^{-1} \int_{-M}^M \exp(-itx) F[f](t) dt,$$

$$S^*[f](x) = \sup_{M > 0} |S_M[f](x)|.$$

Recall that if $f \in L_p(R)$, $p \in [1, 2]$, then operators F, F^* are well defined; for the values $p > 1$, $f \in L_p$ are well defined the operators s_M, s^*, S_M, S^* .

We introduce also for arbitrary $\psi(\cdot) \in U\Psi$, $\text{supp } \psi \supset (1, 2]$, $\psi_1(p) = \psi(p/(p-1))$, for $s = \text{const} \in (1, \infty)$, $\psi(\cdot) \in U\psi$, $\text{supp } \psi \supset (1, s)$

$$\psi_{(s)}(p) = \psi(sp/(s-p)); p = \infty \Rightarrow p/(p-1) = +\infty;$$

for $\psi \in U\Psi$, $\text{supp } \psi \supset [1, s/(s-1))$,

$$\psi^{(s)}(p) = \psi[ps/(s-1)/(p+s/(s-1))].$$

Let $\lambda, \gamma = \text{const} \geq 0$; we denote for $\psi \in U\Psi(1, \infty)$

$$\psi_{\lambda, \gamma}(p) = p^{\lambda+\gamma} \psi(p) (p-1)^{-\gamma}.$$

It is easy to verify that if $\psi \in EX\Psi$, then $\psi_{\lambda, \gamma} \in EX\Psi$.

Let X, Y be a two Banach spaces and let $F : X \rightarrow Y$ be a operator (not necessary linear or sublinear) defined on the space X with values in Y . The operator F is said to be bounded from the space X into the space Y , notation:

$$\|F\| [X \rightarrow Y] < \infty,$$

if for arbitrary $f \in X \Rightarrow \|F[f]\|_Y \leq C \cdot \|f\|_X$.

Theorem 9. Let $\psi \in U\Psi, (1, 2] \subset \text{supp } \psi$. The operator F is bounded from the space $G(\psi)$ into the space $G(\psi_1)$:

$$\|F\| [G(\psi) \rightarrow G(\psi_1)] < \infty.$$

Proof. We will use the classical result of Hardy - Littlewood - Young:

$$|F[f]|_{p/(p-1)} \leq C |f|_p, p \in (1, 2].$$

Here C is an absolute constant.

If $f \in G(\psi)$, then $|f|_p \leq \|f\|G \cdot \psi(p)$, therefore

$$|F[f]|_p \leq \psi(p/(p-1)) \|f\|G(\psi) = \psi_1(p) \|f\|G(\psi).$$

Theorem 10. Let $X = [-\pi, \pi]$, $\psi \in U\Psi$, $\text{supp } \psi \supset (1, \infty)$. We assert that

$$\sup_{M \geq 1} \|s_M\| [G(\psi) \rightarrow G(\psi_{1,1})] < \infty.$$

Proof. Now we use the well-known result of M. Riesz:

$$\|s_M[f]\| [L_p \rightarrow L_p] \leq Cp^2/(p-1), \quad p \in (1, \infty).$$

with absolute constant C . If $f \in G(\psi)$, then $|f|_p \leq$

$$\psi(p) \|f\|G(\psi), \quad |s_M|_p \leq Cp^2 |f|G(\psi)/(p-1) = C \|f\|G(\psi) \cdot \psi_{1,1}(p).$$

Corollary 3. Assume in addition to the conditions of theorem 10 that $\text{supp } \psi \subset (a, b)$ for some $a = \text{const} > 1, a < b = \text{const} < \infty$. Then

$$\psi_{1,1}(p) \asymp \psi(p), \quad p \in (a, b).$$

Therefore, in this case

$$\sup_{M \geq 1} \|s_M\| [G(\psi) \rightarrow G(\psi)] < \infty.$$

However, this assertion does not mean that $\forall f \in G(\psi) \Rightarrow$

$$\lim_{M \rightarrow \infty} \|s_M[f] - f\|G(\psi) = 0;$$

see counterexamples further. If $\nu(\cdot) \in U\Psi$, $\nu < \psi_{1,1}$, $f \in G(\psi)$, then

$$\lim_{M \rightarrow \infty} \|s_M[f] - f\|G(\nu) = 0,$$

i.e. the sequence $s_M[f]$ convergent to the function f in the $G(\nu)$ sense.

At the same assertion is true if $f \in G^0(\psi)$.

The assertion analogous to the assertion of theorem 10 is true for the maximal Fourier's operator s^* , Fourier transform S_M and maximal Fourier transform S^* etc.

Namely, in [13], p. 163 is proved that $\forall f \in L_p, p \in (1, 2] \quad |F^*[f]|_p \leq Cp^4(p-1)^{-2} |f|_p$. Following,

$$\|F^*\| [G(\psi) \rightarrow G(\psi_{2,2})] < \infty.$$

Let us show the exactness of theorem 9. Let $f(x) = f_{a,b}(x) = |x|^{-1/b}, |x| \in (0, 1); f(x) = |x|^{-1/a}, |x| \geq 1; G = G(a, b; 1/a, 1/b), G' = G(b/(b-1), a/(a-1), (b-1)/b, (a-1)/a)$; then $f \in G$. It is easy to calculate that $F[f_{a,b}](t) \asymp f_{b/(b-1), a/(a-1)}(t)$, $t \in R$, so

$$F[f_{a,b}] \in G' \setminus G^0.$$

This example is true even in the case $a = 1$; then $a/(a-1) + \infty$.

For the Fourier series $\sum_n c(n) \exp(inx)$ it is well known (on the basis of Riesz's theorem) that

$$f \in L_p[-\pi, \pi], \exists p > 1 \Rightarrow \lim_{M \rightarrow \infty} |s_M[f] - f|_p = 0.$$

This fact is true also in the Orlicz's spaces with N -function satisfying the so-called $\Delta_2 \cap \nabla_2$ conditions ([6], p. 196 - 197). Conversely, in the exponential Orlicz's spaces there exist functions f , belonging to this spaces but such that Fourier series (or integrals) does not convergent to f in the Orlicz's norm sense [5]. Analogously, this effect appears in $G(\psi)$ spaces.

Lemma 4. *Let $\psi \in EX\Psi$, $X = [-\pi, \pi]$. There exists a function $f \in G(\psi)$ for which the Fourier series does not convergence in $G(\psi)$ norm to the function f .*

Proof. Since $\psi \in EX\Psi$, there exists a function $f : X \rightarrow R$ for which $|f|_p \asymp \psi(p)$, $p \in (a, b)$; then $f \in G \setminus G^0(\psi)$. Assume conversely, i.e.

$$\lim_{M \rightarrow \infty} ||s_M[f] - f||_{G(\psi)} = 0.$$

Since the trigonometrical system is bounded, this means that $f \in G^0$, in contradiction.

8 Martingales.

Let (f_n, F_n) be a martingale, i.e. a monotonically non-decreasing sequence of F_n -sigma-subalgebras Σ and F_n measurable functions f_n such that $\mathbf{E}f_{n+1}/F_n = f_n$.

In this section we will use the probabilistic notations

$$\mathbf{E}f = \int_X f(x) \mu(dx), \quad |f|_p = \mathbf{E}^{1/p} |f|^p$$

and notation $\mathbf{E}f/F$ for the conditional expectation.

The L_p -theory of conditional expectations and theory of martingales in the case $\mu(X) = \infty$ and some applications see, for example, in the book [7], pp. 330 - 347.

The Orlicz's norm estimates for martingales are used in moderne non-parametrical statistics, for example, in the so-called regression problem ([10], [11], [12]) etc. Namely, let us consider the following problem. Given: the observation of a view

$$\xi(i) = g(z(i)) + \epsilon(i), \quad i = 1, 2, \dots,$$

where $g(\cdot)$ is unknown estimated function, $\{\epsilon(i)\}$ is the errors of measurements and may be an independent random variables or martingal differences, $\{z(i)\}$ is some dense set in a metric space (Z, ρ) with Borel measure $\nu : z(i) \in Z$.

Let $\{\phi_k(z)\}$ be some complete orthonormal sequence of functions, for example, classical trigonometrical sequence, Legendre or Hermit polynomials etc. Put

$$c_k(n) = n^{-1} \sum_{i=1}^n \phi_k(z(i)), \quad \tau(N) = \tau(N, n) = \sum_{k=N+1}^{2N} (c_k(n))^2,$$

$$M = \operatorname{argmin}_{n \in [1, n/3]} \tau(N), \quad f_n(z) = \sum_{k=1}^M c_k(n) \phi_k(z).$$

By the investigation of confidence interval for $||f_n - f||$ are used the exponential bounds for polynomial martingales.

The next facts about martingales in the unbounded case $\mu(X) = \infty$ either there are in [7], p. 347 - 351, or are simple generalization of classical results in the case $\mu(X) = 1$ ([8], [9]).

1. Let the martingale (f_n, F_n) be a non - negative, $c, d = \text{const}, 0 < c < d < \infty$ and let for some $p \geq 1$ $\sup_n |f_n|_p < \infty$. Denote by $\nu = \nu(c, d)$ the number of upcrossing of interval (c, d) by the (random) sequence $\{f_n\}$. Then

$$\mathbf{E}\nu \leq (d - c)^{-p} \left[2^{p-1} \sup_n |f_n|_p^p + 2^{p-1} c^p + (d - c)^p \right].$$

2. Almost everywhere convergence. If for some $p \geq 1$ $\sup_n |f_n|_p < \infty$, then $\exists f_\infty(x) = \lim_{n \rightarrow \infty} f_n(x) \pmod{\mu}$, $|f_\infty|_p < \infty$.

3. Convergence in L_p norms. If $\exists p > 1 \Rightarrow \sup_n |f_n|_p < \infty$, then

$$\lim_{n \rightarrow \infty} |f_n - f_\infty|_p = 0.$$

4. Doob's inequality: $p > 1 \Rightarrow$

$$p > 1 \Rightarrow \left| \sup_n f_n \right|_p \leq \sup_n [|f_n|_p]^{p/(p-1)}.$$

In the bounded case $\mu(X) = 1$ the convergence of martingale (*mod* μ) is true under (sufficient) condition $\sup_n |f_n|_1 < \infty$; let us show here that in unbounded case ($\mu(X) = \infty$) our condition is unimproved. Namely, we consider the sequence of independent identically distributed functions $h_j = h_j(x)$ such that for some $p \geq 1$

$$|h_j|_p < \infty; \forall s \neq p, s \geq 1 \Rightarrow |h_j|_s = \infty.$$

Put

$$f_n(x) = \sum_{j=1}^n 2^{-j} h_j(x), \quad F_n = \sigma\{h_j, j \leq n\};$$

then the convergence $f_n(\cdot) \pmod{\mu}$ is true, despite $\forall s \neq p |f_n|_s = \infty$.

It is proved in the book [10], p. 252, see also [11] that if in some Orlicz's space $Or(X, \Sigma, \mu; N) = Or(N)$, with $\mu(X) = 1$ and with the N - Orlicz's function satisfying $\Delta_2 \cap \nabla_2$ condition the martingale $\{f_n\}$ is bounded:

$$\sup_n ||f_n||_{Or(N)} < \infty,$$

then the martingale $\{f_n\}$ convergent in the correspondent Orlicz's norm:

$$\lim_{n \rightarrow \infty} ||f_n - f_\infty||_{Or(N)} = 0.$$

In the article [12] is showed that in the *exponential* Orlicz's spaces $Or(N)$ the $Or(N)$ bounded martingale may divergent. Let us prove that in the $Or(N)$ spaces is the same case.

Lemma 5. Let $\psi \in EX\Psi$, so that $\psi(p) \asymp |f|_p$, and let the σ - algebra $\sigma(f)$ be an union of finite σ - algebras:

$$\sigma(f) = \cup_{n=1}^{\infty} \sigma_n, \quad \text{card}(\sigma_n) < \infty$$

with finite subsets:

$$\forall A \in \sigma_n, A \neq X \Rightarrow \mu(A) < \infty.$$

Then there exists a bounded but divergent in $G(\psi)$ - sense martingale

$$(f_n, F_n) : \sup_n \|f_n\|G(\psi) < \infty, \overline{\lim}_{n \rightarrow \infty} \|f_n - f_\infty\|G(\psi) > 0.$$

Proof. Let us consider some function $f \in G(\psi) \setminus G^0(\psi)$. Put $F_n = \sigma_n$, $f_n = \mathbf{E}f/F_n$; then (f_n, F_n) is a (regular) bounded martingale:

$$\sup_n \|f_n\|G = \sup_{p \in (a,b)} |f_n|_p / \psi(p) \leq \sup_{p \in (a,b)} |f|_p / \psi(p) = \|f\|G < \infty;$$

we used the Iensen inequality $|f_n|_p \leq |f|_p$.

Since the sigma - algebras σ_n are finite, $f_n \in G^0(\psi)$. Suppose $\|f_n - f\|G \rightarrow 0$, $n \rightarrow \infty$, then $f \in G^0$, in contradiction with choosing f .

Theorem 11. Let (f_n, F_n) be a martingale, $\psi \in U\Psi$,

$$\sup_n \|f_n\|G(\psi) < \infty.$$

Then

$$\mathbf{A.} \quad \|\sup_n f_n\|G(\psi_{0,1}) < \infty.$$

Assume in addition that $\text{supp } \psi = (a, b)$, $1 < a < b \leq \infty$. Then $\forall \nu \in U(\psi)$, $\nu \ll \psi_{0,1}$

$$\mathbf{B.} \quad \lim_{n \rightarrow \infty} \|f_n - f_\infty\|G(\nu) = 0.$$

Proof use the Doob's inequality and is the same as in theorem 8 and may be omitted.

For example, let (f_n, F_n) be a martingale, $1 \leq a < b \leq \infty$, $\sup_n \|f_n\|G(a, b; \alpha, \beta) < \infty$. Then in the case $a > 1$ is true the following implication

$$\|\sup_n |f_n|\|G(a, b; \alpha, \beta) < \infty; \forall \Delta > 0 \Rightarrow$$

$$\lim_{n \rightarrow \infty} \|f_n - f_\infty\|G(a, b; \alpha + \Delta, \beta + \Delta[I(b < \infty) - I(b = \infty)]) = 0;$$

if $a = 1$, then

$$\|\sup_n |f_n|\|G(1, b; \alpha + 1, \beta) < \infty; \forall \Delta > 0 \Rightarrow$$

$$\lim_{n \rightarrow \infty} \|f_n - f_\infty\|G(1, b; \alpha + 1 + \Delta, \beta + \Delta[I(b < \infty) - I(b = \infty)]) = 0.$$

It is clear that the convergence $f_n \rightarrow f_\infty$ in the norm $G(a, b; \alpha, \beta)$ is true also in the case $f_\infty \in G^0(a, b; \alpha, \beta)$.

9 Operators.

In this section we assume that there is a measurable space (X, Σ, μ) and Q is an operator not necessary linear or sublinear defined on the set $\cap_{p \in (a,b)} L_p(X, \mu)$, $1 \leq a = \text{const} < b = \text{const} \leq \infty$ and taking values in the set $\cap_{p \in (c,d)} L_p(X, \mu)$. We will investigate the problem of boundedness of operator Q from some space $G(X, \psi)$ into some *another* space $G(X, \nu)$.

The case of Orlicz spaces and certain singular operators was consider in many publications; see, for example, [18], [19], [20].

At first we consider the regular operators.

1. Define a multiplicative operator

$$Q_f[g](x) = f(x) \cdot g(x).$$

Assume that $f \in L_s$ for some $s = \text{const} > 1$ and denote $t = t(s) = s/(s-1)$. As long as

$$|Q_f[g]|_r \leq |f|_s \cdot |g|_{rt/(r+t)}, \quad r < s,$$

we conclude: if $\text{supp } \psi \supset (t(s), \infty)$, then

$$||Q_f||[G(\psi) \rightarrow G(\psi_{(s)})] < |f|_s, \quad \psi_{(s)}(p) = \psi(ps/(s-p)).$$

2. We consider now the convolution operator (again regular)

$$\text{Con}_f[g](x) = f * g(x) = \int_X g(xy^{-1}) f(y) \mu(dy),$$

where X is unimodular Lie's group, μ is its Haar measure. Assume that $f \in L_s(X, \mu)$ for some $s = \text{const} > 1$. Using the classical Young inequality

$$|f * g|_r \leq C(r, s) |f|_s \cdot |g|_{rt(s)/(r+t(s))}, \quad r > s, C(r, s) < 1,$$

we observe that

$$||\text{Con}_f||[G(\psi) \rightarrow (\psi^{(s)})] \leq |f|_s.$$

For example, if $\min(\alpha, \beta) > 0$, then

$$||\text{Con}_f||[G(1, \infty; \alpha, -\beta) \rightarrow G(s, \infty; \alpha, 0)] \leq C(\alpha, \beta, s) |f|_s, \quad s > 1.$$

3. Finally we consider some classical singular operators. Assume that the operator Q satisfies the following condition: for some $\lambda, \gamma = \text{const} \geq 0$ and $\forall p \in (1, \infty)$

$$|Q[f]|_p \leq C |f|_p p^{\lambda+\gamma} (p-1)^{-\gamma}. \quad (8.1)$$

There are many singular operators satisfying this condition, for instance, Hilbert's operator: $X = (-\pi, \pi)$ (or, analogously, $X = R$),

$$H[f](x) = \lim_{\epsilon \rightarrow 0+} H_\epsilon[f](x),$$

$$H_\epsilon[f](x) = (2\pi)^{-1} \int_{\epsilon \leq |y| \leq \pi} [f(x-y)/\tan(y/2)] dy, \quad \lambda = \gamma = 1;$$

maximal Hilbert's operator

$$H^*[f](x) = \sup_{\epsilon \in (0,1)} |H_\epsilon[f](x)|, \quad \lambda = 1, \gamma = 2;$$

operators of Caldron - Zygmund: $\lambda = \gamma = 1$, Karleson - Hunt: $s^*, S^*; \lambda = 1, \gamma = 3$; maximal, in particular, maximal Fourier, operators, for example,

$$Q[f](x) \stackrel{\text{def}}{=} \sup_{M>0} \left| \int_R f(t) [\sin(M(x-t))/(x-t)] dt \right| : \quad \lambda = \gamma = 2;$$

pseudodifferential operators ([15], p. 143): $\lambda = 1 = \gamma$, oscillating operators ([14], p. 379 - 381) etc.

The following result is obvious.

Theorem 12. *Let $\psi \in U\Psi$, $\text{supp } \psi = (1, \infty)$. Assume that the operator Q satisfies the condition (8.1). Then*

$$\|Q\| [G(\psi) \rightarrow G(\psi_{\lambda, \gamma})] < \infty.$$

Let us consider examples. Assume again that the operator Q satisfies the condition (8.1). Then Q is bounded as operator from the space $G(a, b; \alpha, \beta)$ into the space $G(a, b; \alpha_1, \beta_1)$, where at $1 < a < b < \infty \Rightarrow \alpha_1 = \alpha, \beta_1 = \beta$; in the case $a = 1, b < \infty \Rightarrow \alpha_1 = \alpha + \gamma, \beta_1 = \beta$; if $a > 1, b = \infty$ then $\alpha_1 = \alpha, \beta_1 = \beta + \lambda$; ultimately, for $a = 1, b = \infty$ we obtain: $\alpha_1 = \alpha + \gamma, \beta_1 = \beta + \lambda$.

Now we show the exactness of estimations of theorem 12. Consider at first the singular Hilbert operator for the functions defined on the set $(-\pi, \pi)$.

Put now

$$f(x) = f_d(x) = \sum_{n=2}^{\infty} n^{-1} \log^d n \sin(nx), \quad d \geq 0.$$

then (see [16], p. 184; [17], p. 116)) $|f(x)| \asymp (2 + |\log(|x|)|)^d$, $|f|_p \asymp p^d, p \in [1, \infty), x \in [-\pi, \pi] \setminus \{0\}$;

$$CH[f](x) = \sum_{n=2}^{\infty} n^{-1} \log^d n \cos(nx),$$

$$H[f](x) \asymp (2 + |\log(|x|)|)^{d+1}, \quad |H[f]|_p \asymp p^{d+1}.$$

Considering the examples $d \in (0, 1), g = g_d(x) =$

$$\sum_{n=1}^{\infty} n^{d-1} \sin(nx), \quad CH[g] = \sum_{n=1}^{\infty} n^{d-1} \cos(nx),$$

we can see that $|g(x)| \asymp |H[g]|(x), x \in R \setminus \{0\}$, and following $|g|_p \asymp |H[g]|_p, p \in (1, \infty)$.

We can built more general examples considering the functions of a view

$$f(x) = \sum_{n=2}^{\infty} n^{d-1} L(n) \sin(nx),$$

where $L(n)$ is some slowly varying as $n \rightarrow \infty$ function. See [17], p. 187 - 188.

The case of Hilbert's transform on the real axis is investigated analogously. Namely, consider the functions

$$f(x) = \int_3^{\infty} t^{d-1} \sin(tx) dt, \quad d \in (0, 1),$$

then (see [17], p.117) $CH[f](x) =$

$$\int_3^{\infty} t^{d-1} \cos(tx) dt, \quad |H[f](x)| \asymp |f(x)| \asymp f_{1/d,1}(x),$$

following,

$$H[f](\cdot), f(\cdot) \in G \setminus G^o(1, 1/d; 1, d).$$

Analogously, considering the example

$$f(x) = \int_3^\infty t^{-1} \sin(tx) dx, \quad |f(x)| \asymp f_{\infty,1}(x),$$

$x \in R \setminus \{0\}$, we observe that $|H[f](x)| \asymp |\log |x||$, $|x| \leq 1/2$;

$$f(\cdot) \in G \setminus G^o(1, \infty; 1, 0), \quad |CH[f](x)| \sim |\log |x||, \quad x \rightarrow 0;$$

$$|H[f](x)| \asymp |x|^{-1}, \quad |x| \geq 1/2,$$

so that $H[f](\cdot) \in G \setminus G^o(1, \infty; 1, 1)$,

Acknowledgement. We are very grateful for support and attention to prof. V.Fonf, L.Beresansky, and M.Lin (Beer - Sheva, Israel). This investigation was partially supported by ISF (Israel Science Foundation), grant N^o 139/03.

References.

1. Krasnoselsky M.A., Rutitsky Ya.B. Convex functions and Orlicz's Spaces. P. Noordhoff LTD, The Netherlands, 1961, Groningen.
2. Rao M.M., Ren Z.D. Theory of Orlicz Spaces. - New York, Basel. Marcel Dekker, 1991. - 449 p.
3. Davis H.W., Murray F.J., Weber J.K. Families of L_p -spaces with inductive and projective topologies. // Pacific J. Math. - 1970 - v. 34, p. 619 - 638.
4. Steigenwalt M.S. and While A.J. Some function spaces related to L_p . // Proc. London Math. Soc. - 1971. - 22, p. 137 - 163.
5. Ostrovsky E., Sirota L. Fourier Transforms in Exponential Rearrangement Invariant Spaces. // Electronic Publ., arXiv:Math., FA/040639, v.1, - 20.6.2004.
6. Rao M.M., Ren Z.D. Application of Orlicz Spaces. - New York, Basel. Marcel Dekker, 2002. - 475 p.
7. Rao M.M. Measure Theory and Integration. - New York, Basel, Marcel Dekker, second edition, - 2004. - 781 p.
8. Hall P., Heyde C.C. Martingale Limit Theorems and its Applications. - USA, New York, Academic Press Inc., 1980, - 473 p.
9. Doob J.L. Stochastic Processes. - New York, John Wiley, - 1953, 671 p.
10. Neveu J. Discrete - Parameter Martingales. - Amsterdam - Oxford - New York, North - Holland Publ. Comp., 1975. - 385 p.
11. Peshkir G. Maximal Inequalities of Kahane - Khintchine type in Orlicz Spaces. // Preprint Series 33, Inst. of Math., 1992, University of Aarhus (Denmark). - 1992. - 1 - 44 p.
12. Ostrovsky E. Exponential Orlicz Spaces: new Norms and Applications. // Electronic Publ., arXiv/FA/0406534, v. 1, - 25.06.2004.
13. Juan Arias de Reyna. Pointwise Convergence Fourier Series. - New York, Lect. Notes in Math. 2004. - 286 p.
14. Stein E.M. Singular Integrals and Differentiability Properties of Functions. - Princeton, New York. University Press. 1970. - 572 p.
15. Taylor M. Pseudodifferential Operators. - Princeton, New Jersey, University Press. 1981. - 473 p.
16. Zygmund A. Trigonometrical Series. Volume 1. Cambridge University Press, 1959.
17. Seneta E. Regularly Varying Functions. 1985, Moscow edition.
18. Bongioanni B., Forzani L., Harboure E. Weak type and restricted weak type (p,p) operators in Orlicz spaces. (2002/2003). Real Analysis Exchange, Vol. 28(2), pp. 381 - 394.
19. Harboure E., Salinas O. and Viviani B. Orlicz Roundedness for Certain Classical Operators. Colloquium Mathematicum 2002, 91 (No 2), pp. 263 - 282.
20. Cianchi A. Hardy inequalities in Orlicz Spaces. Trans. AMS, **351**, No 6 (1999), 2456 - 2478.
21. Krein S.G., Petunin Yu., and Semenov E.M. Interpolation of linear operators. AMS, 1982.

Ostrovsky Eugene, Sirota Leonid.
Moment Banach Spaces: theory and applications.
Abstract.

In this article we introduce and investigate a new class of Banach spaces, so - called moment spaces, i.e. which are based on the classical $L(p)$ spaces, study their properties: separability, reflexivity, embedding theorems etc., and describe some applications to the theory of Fourier series and transform, theory of martingales, and singular integral operators.

Key words: Banach spaces, moments, Fourier series and transform, martingales, singular operators.

References: 21 works.